DISCRETE CDF 9/7 WAVELET TRANSFORM FOR FINITE-LENGTH SIGNALS

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Abstract

Wavelets and a discrete wavelet transform were originally defined on an infinite interval. Therefore if they are applied to a bounded interval the boundary effects often occur. These effects can be eliminated by using wavelets adapted to the unit interval. In our contribution, we propose a construction of Cohen-Daubechies-Feauveau 9/7 wavelets on the unit interval.

1 Introduction

Originally, wavelets were constructed on the whole real line and the discrete wavelet transform was defined for an infinite signal. In many applications, wavelets on a bounded domain are used and the discrete wavelet transform is applied to a finite signal. The starting point of the construction of wavelets on a general domain is the construction of wavelets on the interval. It insists in retaining basis functions from $L^2(\mathbb{R})$ whose support is contained in the interval and suitable adaptation near the boundaries. Then the filters for the discrete wavelet transform are retained for the interior part of the signal and special filters are used near the boundaries. Spline wavelets and orthogonal wavelets on the interval were constructed in many papers [4, 5, 7, 8, 9]. General biorthogonal wavelets on the interval were designed in [1].

In this paper, we focus on Cohen-Daubechies-Feauveau wavelets with the low-pass filters of the length 9 and 7 (CDF 9/7) designed in [3], which are the most popular wavelets for a wavelet-based image compression. Since the CDF 9/7 wavelet basis adapted to the interval according to [1] is badly conditioned, we propose an adaptation of this wavelet basis to the interval, which leads to a better conditioned wavelet basis. The condition number of the basis plays an important role. It guaranties the stability of the computation and it affects the constants in error estimates.

2 Wavelet Bases

First we introduce concepts from the wavelet theory and notations. We consider the domain $\Omega \subset \mathbb{R}^d$ and the space $L^2(\Omega)$ and we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the L^2-inner product and the L^2-norm, respectively. Let $\mathcal{J}$ be at most countable index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j,k)$, $j,k \in \mathbb{Z}$, where $|\lambda| := j$ is a *scale* or a *level*. Let

$$l^2(\mathcal{J}) := \left\{ v = \{v_\lambda\}_{\lambda \in \mathcal{J}} : v_\lambda \in \mathbb{R} \text{ and } \sum_{\lambda \in \mathcal{J}} |v_\lambda|^2 < \infty \right\}. \tag{1}$$

A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\} \subset L^2(\Omega)$ is called a *wavelet basis* of $L^2(\Omega)$, if

i) Ψ is a *Riesz basis* for $L^2(\Omega)$, i.e. the closure of the linear span of Ψ is complete in $L^2(\Omega)$ and there exist constants $c, C \in (0,\infty)$ such that

$$c\,\|\mathrm{b}\|_{l^2(\mathcal{J})} \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\| \leq C\,\|\mathrm{b}\|_{l^2(\mathcal{J})}, \quad \mathrm{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{2}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies } (2)\}$, $C_\psi := \inf\{C : C \text{ satisfies } (2)\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi / c_\psi$ is called the *condition number* of Ψ.

ii) The functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \leq C2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap in any point $x \in \Omega$.

By the Riesz representation theorem, there exists a unique family $\tilde{\Psi} = \left\{\tilde{\psi}_\lambda, \lambda \in \mathcal{J}\right\}$ in $L^2(\Omega)$ biorthogonal to Ψ, i.e. $\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle = \delta_{i,j}\delta_{k,l}$, for all $(i,k), (j,l) \in \mathcal{J}$. This family is also a Riesz basis for $L^2(\Omega)$. The basis Ψ is called a *primal* wavelet basis, $\tilde{\Psi}$ is called a *dual* wavelet basis.

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis. A sequence $S = \{S_j\}_{j \geq j_0}$ of closed linear subspaces $S_j \subset L^2(\Omega)$ is called a *multiresolution* or *multiscale analysis*, if $S_{j_0} \subset S_{j_0+1} \subset \ldots \subset S_j \subset S_{j+1} \subset \ldots L^2(\Omega)$ and $\cup_{j \geq j_0} S_j$ is complete in $L^2(\Omega)$.

The nestedness and the closedness of the multiresolution analysis implies the existence of the *complement spaces* W_j such that $S_{j+1} = S_j \oplus W_j$. We assume that S_j and W_j are spanned by sets of basis functions $\Phi_j := \{\phi_{j,k}, k \in \mathcal{I}_j\}$ and $\Psi_j := \{\psi_{j,k}, k \in \mathcal{J}_j\}$. Here, $\mathcal{I}_j$ and $\mathcal{J}_j$ are at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The wavelet basis of $L^2(\Omega)$ is obtained by $\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j$. The dual wavelet system $\tilde{\Psi}$ generates a dual multiresolution analysis $\tilde{S}$ with dual scaling bases $\tilde{\Phi}_j$ and dual single-scale wavelet bases $\tilde{\Psi}_j$.

Polynomial exactness of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that $\mathbb{P}_{N-1}(\Omega) \subset S_j$ and $\mathbb{P}_{\tilde{N}-1}(\Omega) \subset \tilde{S}_j$, where $\mathbb{P}_m(\Omega)$ is the space of all algebraic polynomials on Ω of degree less than or equal to m.

From the definition of spaces S_j and W_j it follows that there exist *refinement matrices* $\mathbf{M}_{j,0}$, $\tilde{\mathbf{M}}_{j,0}$, $\mathbf{M}_{j,1}$, and $\tilde{\mathbf{M}}_{j,1}$ such that

$$\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}, \ \tilde{\Phi}_j = \tilde{\mathbf{M}}_{j,0}^T \tilde{\Phi}_{j+1}, \ \Psi_j = \mathbf{M}_{j,1}^T \Phi_{j+1}, \ \tilde{\Psi}_j = \tilde{\mathbf{M}}_{j,1}^T \tilde{\Phi}_{j+1}. \tag{3}$$

The *discrete wavelet transform* consists of applying $\mathbf{M}_j^T = (\mathbf{M}_{j,0} \ \mathbf{M}_{j,1})^T$.

3 Wavelet basis in $L^2(\mathbb{R})$

Wavelet bases on the interval are derived from the wavelet bases for the space $L^2(\mathbb{R})$. A function $\psi \in L^2(\mathbb{R})$ is called a *wavelet* for the space $L^2(\mathbb{R})$ if the family $\Psi := \{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$, where $\psi_{j,k}(x) := 2^{j/2}\psi\left(2^j x - k\right)$, $x \in \mathbb{R}$, is a Riesz basis in $L^2(\mathbb{R})$. The functions $\psi_{j,k}$ are also called wavelets.

Let S_j be the closure of the span of the set $\{\psi_{l,k}, l \leq j, k \in \mathbb{Z}\}$ and let us suppose that there exists a function ϕ such that $\Phi_j := \{\phi_{j,k}, k \in \mathbb{Z}\}$, $\phi_{j,k}(x) := \phi\left(2^j x - k\right)$, $x \in \mathbb{R}$, is a Riesz basis of S_j. Functions ϕ and $\phi_{j,k}$ are called *scaling functions*. Then there exists a sequence $\{h_k\}_{k \in \mathbb{Z}}$ such that

$$\phi(x) = \sum_{k \in \mathbb{Z}} h_k \phi(2x - k), \quad x \in \mathbb{R}. \tag{4}$$

This equation is called a *refinement* or a *scaling equation* and the coefficients h_k are known as *scaling* or *refinement coefficients*. These coefficients form filters used in the discrete wavelet transform. The dual scaling basis $\tilde{\Psi}$ is also formed by translations and dilations of one function $\tilde{\phi}$. A dual scaling equation and dual scaling coefficients $\tilde{h}_k$ are defined in a similar way as in the primal side.

We define *wavelet coefficients* as $g_n := (-1)^n \tilde{h}_{1-n}$ and $\tilde{g}_n := (-1)^n h_{1-n}$. Wavelets are then given by

$$\psi(x) = \sum_{n \in \mathbb{Z}} g_n \phi(2x - n), \quad \tilde{\psi}(x) = \sum_{n \in \mathbb{Z}} \tilde{g}_n \tilde{\phi}(2x - n), \quad x \in \mathbb{R}. \tag{5}$$

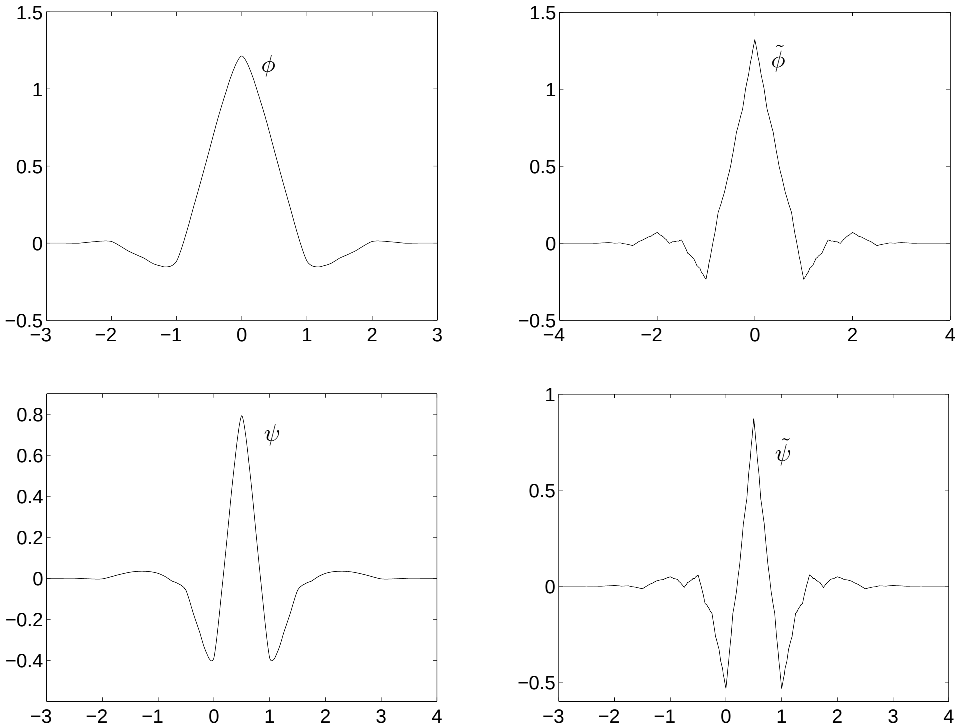

Figure 1: Cohen-Daubechies-Feauveau 9/7 scaling functions and wavelets.

In this paper, we focus on Cohen-Daubechies-Feauveau 9/7 wavelets from [3]. Their polynomial exactness is four both on the primal and the dual side. Figure 1 shows the graphs of scaling functions and wavelets.

4 Construction of a Primal Scaling Basis

We consider the scaling function from [3], its support is $[-3,3]$. Therefore the support of $\phi_{j,k}$ is contained for some k in the interval $[0,1]$ only for $j \geq 3$. For this reason the coarsest level of the primal scaling basis is three and in the following we always assume that $j \geq 3$. We define inner scaling functions as translations and dilations of the primal scaling function ϕ from [3]:

$$\phi_{j,k}(x) := 2^{j/2}\phi\left(2^j x - k\right), \quad x \in [0,1], \quad k = 4, \dots 2^j - 2. \tag{6}$$

We define the boundary scaling functions in order to preserve the polynomial exactness of order four by the similar way as in [1, 4]:

$$\phi_{j,k}(x) := 2^{j/2}\sum_{l=-5}^{-1} M_{k,l}\phi\left(2^j x - l\right), \quad x \in [0,1], \quad k = 0, \dots, 3,$$

where

$$M_{k,l} := \int_{-\infty}^{\infty} p_k(x)\,\tilde{\phi}(x-l)\,dx, \quad k = 0, \dots, 3, \tag{7}$$

and $p_0, \dots, p_3$ is a basis of $\mathbb{P}_3([0,1])$. As in [8] we choose Bernstein polynomials, because they lead to better conditioned resulting primal scaling basis than monomials. Bersteins polynomials are given by:

$$p_k(x) := c^{-3}\binom{3}{k} x^k (c-x)^{3-k}, \quad x \in \mathbb{R}, \quad k = 0, \dots, 3. \tag{8}$$

The choice of parameter c affects the condition number of the resulting basis. By numerical experiments we found that $c = 3$ is an appropriate choice. The boundary functions at the right boundary are defined to be symmetrical with the left boundary functions:

$$\phi_{j,k}(x) := \phi_{j,2^j+2-k}(1-x), \quad x \in [0,1], \quad k = 2^j - 1, \ldots, 2^j + 2. \tag{9}$$

We normalize primal scaling functions $\phi_{j,k}$ with respect to the L^2-norm.

5 Construction of a Dual Scaling Basis

The desired property of the dual scaling basis $\tilde{\Phi}$ is biorthogonality to Φ and polynomial exactness of order four. Let $\tilde{\phi}$ be a dual scaling function which was designed in [3]. Its support is $[-4, 4]$. We define inner scaling functions as translations and dilations of $\tilde{\phi}$:

$$\theta_{j,k}(x) := 2^{j/2}\tilde{\phi}\left(2^j x - k\right), \quad x \in [0,1], \quad k = 5, \ldots 2^j - 3. \tag{10}$$

There will be two types of basis functions at each boundary. Basis functions of the first type are defined to preserve polynomial exactness:

$$\theta_{j,k}(x) := 2^{j/2} \sum_{l=-7}^{-1} \tilde{M}_{k,l}\ \tilde{\phi}\left(2^j x - l\right), \quad x \in [0,1], \quad k = 0, \ldots, 3,$$

where

$$\tilde{M}_{k,l} := \int_{-\infty}^{\infty} \tilde{p}_k(x)\, \phi(x-l)\, dx, \quad k = 0, \ldots, 3, \tag{11}$$

and $\tilde{p}_0, \ldots, \tilde{p}_3$ are Bernstein polynomials for parameter $c = 1.2$.

We define the basis function of the second type by

$$\theta_{j,4}(x) := 2^{j/2} \sum_{l=2}^{8} \tilde{h}_l \tilde{\phi}\left(2^j x + 2 - l\right), \quad x \in [0,1].$$

Then $\theta_{j,0}$ lies in the linear span of $\left\{\theta_{j+1,k} : k = 0, \ldots, 2^j + 2\right\}$ and therefore the nestedness of multiresolution spaces is preserved.

The boundary functions at the right boundary are defined to be symmetrical with the left boundary functions:

$$\theta_{j,k}(x) := \theta_{j,2^j+2-k}(1-x), \quad x \in [0,1], \quad k = 2^j - 2, \ldots, 2^j + 2. \tag{12}$$

Since the set $\Theta_j := \left\{\theta_{j,k} : k = 0, \ldots, 2^j + 2\right\}$ is not biorthogonal to Φ_j, we derive a new set $\tilde{\Phi}_j$ from Θ_j by biorthogonalization. Let $\mathbf{Q}_j := \left(\langle \phi_{j,k}, \theta_{j,l} \rangle\right)_{k,l=0,\ldots 2^j+2}$. Viewing $\tilde{\Phi}_j$ and Θ_j as column vectors we define $\tilde{\Phi}_j := \mathbf{Q}_j^{-T}\Theta_j$. Then $\tilde{\Phi}_j$ is biorthogonal to Φ_j. The biorthogonalization matrix can be computed by the method from [6].

Due to the length of support of primal scaling functions, the refinement matrix $\mathbf{M}_{j,0}$ has the following structure:

$$\mathbf{M}_{j,0} = \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{A}_j & \\ & & \mathbf{M}_R \end{pmatrix}, \tag{13}$$

where $\mathbf{A}_j$ is a matrix of the size $\left(2^{j+1}-5\right) \times \left(2^j-5\right)$:

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} h_0 & h_1 & h_2 & \dots & h_6 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & h_0 & \dots & h_4 & h_5 & h_6 & 0 & \dots & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & h_0 & \cdots & h_5 & h_6 \end{pmatrix}^T, \tag{14}$$

where $h_0, \dots, h_6$ are the scaling coefficients corresponding to ϕ. The matrices $\mathbf{M}_L$ and $\mathbf{M}_R$ contain boundary filters. They can be found by the same method as in [5]. The refinement matrix $\tilde{\mathbf{M}}_{j,0}$ corresponding to $\tilde{\Phi}$ has the similar structure.

Our next goal is to determine the corresponding wavelets. We follow a general principle called stable completion which was proposed in [2]. We found the initial stable completion by the method from [8].

The condition numbers of constructed bases are:

$$\text{cond}\Phi_9 = 22.3, \quad \text{cond}\tilde{\Phi}_9 = 32.1, \quad \text{cond}\Psi_9 = 14.4, \quad \text{cond}\tilde{\Psi}_9 = 22.2. \tag{15}$$

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