COMPLETE RESPONSE OF SECOND AND HIGHER-ORDER LINEAR ELECTRIC CIRCUITS

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Abstract

The paper is aimed to the proposal for analysis of second and higher-order linear electric circuits with the purpose to find the complete response of the circuits. A very powerful tool for the analysis of these circuits is to transform the circuits directly into the complex frequency domain using the Laplace transform and then apply the techniques (mesh analysis, node analysis) developed for resistor-only circuits to solve them. Applying above mentioned method together with the symbolic computation of MATLAB there is no difficulty in solving second and higher-order circuits and finding the complete response of these circuits.

1 Principles of Finding Complete Response of Second and Higher-order Linear Electric Circuits

To find the complete response of second and higher-order circuit means to solve a dynamic circuit. The dynamic circuit is a circuit that has not only resistors but also dynamic elements (inductors and/or capacitors). Analysis of these circuits is sometimes time-consuming activity, because the equations for such type of circuits take the form of integrodifferential equations.

One of the tools for the analysis of these circuits is to transform the circuit directly into the complex frequency domain using the Laplace transform. The Laplace transform is defined for a function $x(t)$ of a real variable *t* (meaning the time) as [1], [2]

$$\hat{x}(s) = \pounds\left\{x(t)\right\} = \int_0^\infty x(t)e^{-st}\mathrm{d}t\,, \tag{1}$$

where *s* is a complex variable (meaning the complex frequency), and $x(t)$ is zero for all $t < 0$.

The time domain is also called *t*-domain and the complex frequency domain is called *s*-domain.

After transforming a given circuit with its initial conditions into its the *s*-domain equivalent, it can be dealt with it as if it consisted of sources and resistors only, because the passive elements have impedances $R, sL, \frac{1}{sC}$, which can be regarded as generalized resistances.

Two systematic methods can be used to solve the transformed circuit (to find all branch currents and voltages): the mesh analysis method and the node analysis method. As a result of applying one of these methods to the circuit, a set of mesh or node algebraic equations is constituted.

Solving the set of these equations, the *s*-domain solution of the mesh currents/node voltages is obtained. The *s*-domain branch currents are found based on the mesh currents/node voltages and the *s*-domain element voltages can be expressed using voltage-current relationship. To get the *t*-domain solution, which represents the complete response of the given circuit, it is necessary to take the inverse Laplace transform of the *s*-domain solution.

The complete response of the linear electric circuit excited by DC or AC sources can be decomposed into two components. These components are the transient response (the component that dies out as time goes) and the steady state response (the component that survives time like a constant or a sinusoid). Character of the transient response of the circuit is dependent on the characteristic roots i.e. the roots of the characteristic equation. These roots are very important because the behavior of second-order and higher-order circuit is characterized by them.

All the above given steps which must be done to solve second-order and higher-order circuits can be easily executed using MATLAB, especially the symbolic computation.

2 Results

Let us consider the circuit (Figure 1), in which the values of the source voltages, the resistors, the inductor, and the capacitor are: $U_1 = 12\,\mathrm{V}$, $U_2 = 24\,\mathrm{V}$, $R_1 = 10\,\Omega$, $R_2 = 2\,\Omega$, $L = 2\,\mathrm{H}$, $C = 1/4\,\mathrm{F}$, respectively. It is the second-order circuit and it is supposed the switch has been connected to the DC voltage source U_1 for a long time before $t = 0$ when it is flipped to the DC voltage source U_2. Let us solve this circuit for $t \geq 0$.

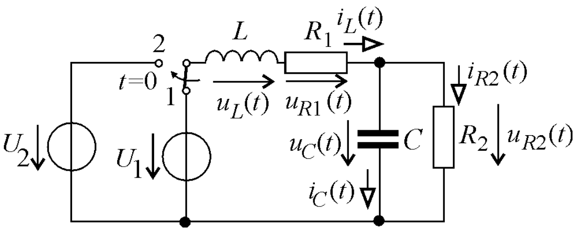

Figure 1: Second-order circuit driven by DC source

Since the circuit is supposed to be in DC steady state, the initial values of the inductor current and the capacitor voltage at $t = 0$ (the initial conditions) can be found to be (see Figure 2 – left):

$$i_L(0) = i_L(0_-) = i_L(0_+) = \frac{U_1}{R_1 + R_2} \quad \text{and} \quad u_C(0) = u_C(0_-) = u_C(0_+) = R_2 \frac{U_1}{R_1 + R_2}. \tag{2}$$

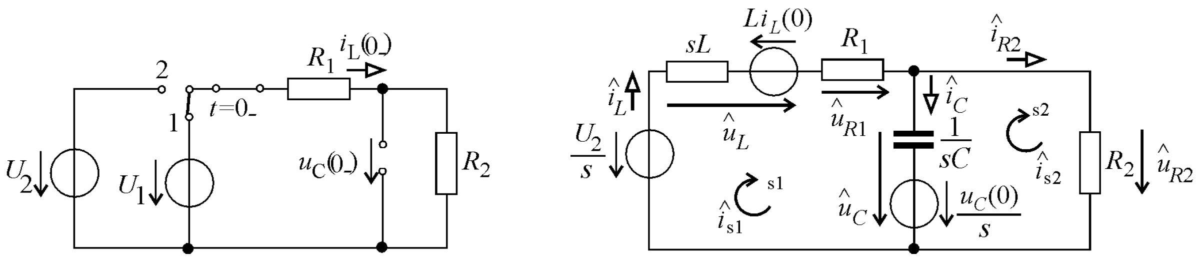

Figure 2: A given circuit in DC steady state (left) and the s-domain equivalent with the initial conditions represented as voltage sources (right)

In Figure 2 (right) is depicted the s-domain equivalent circuit with the initial conditions represented as voltage sources that suits the mesh analysis. For this circuit can be written the mesh equations in matrix – vector form as

$$\begin{bmatrix} R_1 + sL + \dfrac{1}{sC} & -\dfrac{1}{sC} \\ -\dfrac{1}{sC} & R_2 + \dfrac{1}{sC} \end{bmatrix} \begin{bmatrix} \hat{i}_{s1}(s) \\ \hat{i}_{s2}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{U_2}{s} - \dfrac{u_C(0)}{s} + Li_L(0) \\ \dfrac{u_C(0)}{s} \end{bmatrix}. \tag{3}$$

Substituting the parameter values of the circuit elements and the values of the initial conditions yields

$$\begin{bmatrix} 10 + 2s + \dfrac{4}{s} & -\dfrac{4}{s} \\ -\dfrac{4}{s} & 2 + \dfrac{4}{s} \end{bmatrix} \begin{bmatrix} \hat{i}_{s1}(s) \\ \hat{i}_{s2}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{24}{s} - \dfrac{2}{s} + 2 \\ \dfrac{2}{s} \end{bmatrix}. \tag{4}$$

This set of equations is solved to obtained the expression of the transformed mesh currents as

$$\begin{bmatrix} \hat{i}_{s1}(s) \\ \hat{i}_{s2}(s) \end{bmatrix} = \frac{1}{s\left(s^2 + 7s + 12\right)} \begin{bmatrix} s^2 + 13s + 24 \\ s^2 + 7s + 24 \end{bmatrix}. \tag{5}$$

The transformed branch currents expressed in terms of the transformed mesh currents take the following form

$$\begin{bmatrix} \hat{i}_L(s) \\ \hat{i}_C(s) \\ \hat{i}_{R2}(s) \end{bmatrix} = \begin{bmatrix} \hat{i}_{s1}(s) \\ \hat{i}_{s1}(s) - \hat{i}_{s2}(s) \\ \hat{i}_{s2}(s) \end{bmatrix} = \frac{1}{s\left(s^2+7s+12\right)} \begin{bmatrix} s^2+13s+24 \\ 6s \\ s^2+7s+24 \end{bmatrix}. \tag{6}$$

and the transformed voltages across the elements are

$$\begin{aligned} \hat{u}_L(s) &= sL\hat{i}_L(s) - Li_L(0) = 12\frac{s+2}{s^2+7s+12} \\ \hat{u}_C(s) &= \frac{1}{sC}\hat{i}_C(s) + \frac{u_C(0)}{s} = 2\frac{s^2+7s+24}{s\left(s^2+7s+12\right)} \\ \hat{u}_{R1}(s) &= R_1\hat{i}_L(s) = 10\frac{s^2+13s+24}{s\left(s^2+7s+12\right)} \\ \hat{u}_{R2}(s) &= R_2\hat{i}_{R2}(s) = 2\frac{s^2+7s+24}{s\left(s^2+7s+12\right)} \end{aligned} \tag{7}$$

Thus the inverse Laplace transform of the branch currents and the voltage across the elements are obtained as

$$\left.\begin{aligned} i_L(t) &= \pounds^{-1}\left\{\hat{i}_L(s)\right\} = 2 - 3e^{-4t} + 2e^{-3t} \\ i_C(t) &= \pounds^{-1}\left\{\hat{i}_C(s)\right\} = -6e^{-4t} + 6e^{-3t} \\ i_{R2}(t) &= \pounds^{-1}\left\{\hat{i}_{R2}(s)\right\} = 2 + 3e^{-4t} - 4e^{-3t} \\ u_L(t) &= \pounds^{-1}\left\{\hat{u}_L(s)\right\} = 24e^{-4t} - 12e^{-3t} \\ u_C(t) &= \pounds^{-1}\left\{\hat{u}_C(s)\right\} = 4 + 6e^{-4t} - 8e^{-3t} \\ u_{R1}(t) &= \pounds^{-1}\left\{\hat{u}_{R1}(s)\right\} = 20 - 30e^{-4t} + 20e^{-3t} \\ u_{R2}(t) &= \pounds^{-1}\left\{\hat{u}_{R2}(s)\right\} = 4 + 6e^{-4t} - 8e^{-3t} \end{aligned}\right\} \text{ for } t \geq 0 \tag{8}$$

It is obvious that the responses of the circuit for given set of the parameters have the over-damped character (the characteristic roots are two distinct real roots).

All of the above computations for analysis of the circuit (for given parameters) depicted in Figure 1 were done by running MATLAB program. The branch currents and the voltages across the elements with respect to time are shown in Figure 3 and their analytic form is shown in Figure 4.

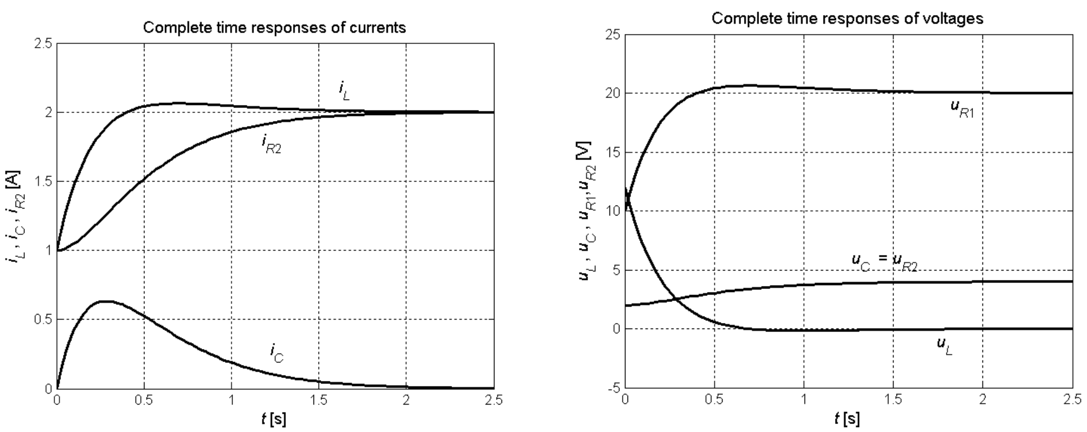

Figure 3: The output currents (left) and voltages (right) of the given circuit

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iL(t) =

                        -3 exp(-4 t) + 2 exp(-3 t) + 2
iC(t) =

                          -6 exp(-4 t) + 6 exp(-3 t)
iR2(t) =

                         3 exp(-4 t) - 4 exp(-3 t) + 2
uL(t) =

                          24 exp(-4 t) - 12 exp(-3 t)
uC(t) =

                         4 + 6 exp(-4 t) - 8 exp(-3 t)
uR1(t) =

                       -30 exp(-4 t) + 20 exp(-3 t) + 20
uR2(t) =

                         4 + 6 exp(-4 t) - 8 exp(-3 t)
>>
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Figure 4: The output currents (left) and voltages (right) of the given circuit expressed in analytic form

References

[1] R.C.Dorf, J.A.Svoboda: *Introduction to Electric Circuits*. John Wiley &Sons, Inc., 2007.
[2] D. Mayer: *Introduction in Theory of Electric Circuits.* SNTL– Prague, 1984 (in Czech).
[3] MATLAB - User's Guide, MathWorks 2009.

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